Eur. Phys. J. C 12, 535–549 (2000)
Digital Object Identifier (DOI) 10.1007/s100529900234

THE EUROPEAN PHYSICAL JOURNAL C

On the trees of quantum fields

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Received: 6 July 1999 / Published online: 10 December 1999

Abstract. The solution of some equations involving functional derivatives is written as a series indexed by planar binary trees. The terms of the series are given by an explicit recursive formula. Some algebraic properties of these series are investigated. Several examples are treated in the case of quantum electrodynamics: the complete fermion and photon propagators, the two-body Green function and the one-body Green function in the presence of an external source, the complete vacuum polarization, the electron self-energy and the irreducible vertex.

1 Introduction

Renormalization theory recently has been revitalized by the discovery of a Hopf algebra that transforms the dreadful combinatorics of renormalization into a mechanical application of the Hopf algebra properties of rooted trees [1–4].

In the companion paper [5], Butcher's theory has been presented as an alternative way to describe the Hopf structure of the algebra of renormalization. A particularly useful aspect of Butcher's approach is that solutions of nonlinear differential equations can be written as a sum over rooted trees.

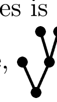
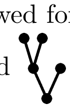
In the present paper, Butcher's strategy is adapted to equations involving functional derivatives that were first proposed by Schwinger [6] and will be called Schwinger equations in the rest of this paper. Schwinger equations are not commonly considered as a useful tool for computation. The purpose of this article is to show that, by using series over planary binary trees, Schwinger equations can be turned into explicit calculation methods.

The series we manipulate are indexed by planar binary trees. So we first present some basic properties of planar binary trees. Then the solution of simple Schwinger equations is written as a sum over planar binary trees, with recursively defined coefficients. To make a comparison with power series, the Schwinger equation would correspond to a differential equation for the sum of the series, whereas the formula we put forward corresponds to a recursive definition of the terms of the series: it does not contain so much information as the Schwinger equation, but it is more explicit if we want to calculate the terms of the series.

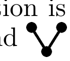
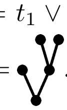
As an example, the full fermion and photon propagators of quantum electrodynamics (QED) are written as a sum over planar binary trees. Other applications are given for QED with an external source, the vacuum polarization, the fermion self-energy and the irreducible vertex.

2 Planar binary trees

In contrast to [5], we do not use rooted trees but planar binary trees. Both can be drawn on a plane, but no permutation of vertices is allowed for planar trees.

As an example,  and  are two different planar trees, although they represent the same rooted tree.

In planar trees, we distinguish two types of vertices: the leaves (which have no children) and the remaining vertices (including the root), which we call internal vertices. In planar binary trees, internal vertices have exactly two children.

We now follow the notation of Loday and Ronco [7]. Planar binary trees have an odd number of vertices. We denote by Y_n the set of planar binary trees with $2n+1$ vertices. If t belongs to Y_n, t has $n+1$ leaves and n internal vertices. The number of elements of Y_n is $(2n)!/(n!(n+1)!)$: the Catalan numbers, which enter many combinatorial problems [8] and should probably be called Ming numbers [9]. If $t_1 \in Y_m$ and $t_2 \in Y_n$, the grafting of t_1 and t_2 is the tree $t \in Y_{m+n+1}$ obtained by putting t_1 on the left of t_2 and by joining the roots of t_1 and t_2 to a new vertex that becomes the root of t. This operation is denoted by $t = t_1 \vee t_2$. For instance, grafting $\bullet$ and  gives $\bullet \vee$  $=$ .

In the companion paper, rooted trees were graded by the number of their vertices. Here, planar binary trees have an odd number of vertices, and it is more natural to grade them differently: for each tree t, we define $|t|$ as the integer n such that $t \in Y_n$. Thus, a tree t has $2|t|+1$ vertices.

An essential property of planar binary trees is that each tree t different from $\bullet$ can be written in a unique way as $t_1 \vee t_2$, where t_1 and t_2 are called the branches of t. Moreover, grafting provides a recursive definition of planar binary trees [10]:

$$Y_{n+1} = \bigcup_{k=0}^{n} Y_k \vee Y_{n-k}, \quad Y_0 = \{\bullet\}. \tag{1}$$

The notation $Y_k \vee Y_{n-k}$ means that all the trees of Y_k are grafted with all the trees of Y_{n-k}.

Planar binary trees have received much attention recently because of their relation to new algebraic structures [10, 11].

3 Schwinger equations

In this section, the solution of a linear Schwinger equation is given as a sum over planar binary trees. But we first introduce the concept of a functional derivative.

A functional $A(\phi)$ is defined loosely as a map sending a distribution ϕ to a complex number (see [12] for details). If ψ is a distribution, the functional derivative of $A(\phi)$ in the direction ψ is defined as the limit for $\epsilon \to 0$ of $(A(\phi+\epsilon\psi) - A(\phi))/\epsilon$. Finally, the functional derivative of $A(\phi)$ with respect to $\phi(x)$, $\delta A(\phi)/\delta\phi(x)$, is defined as the functional derivative of $A(\phi)$ in the direction δ_x, where δ_x is the Dirac function $\delta_x(y) = \delta(y-x)$.

3.1 Examples of functional derivatives

A classical example is $A(\phi) = \int \mathrm{d}x f(x)\phi(x)$, giving easily $\delta A(\phi)/\delta\phi(x) = f(x)$. A further example, that will be useful in the sequel, is $A(\phi) = \int \mathrm{d}x\mathrm{d}y f(x,y)\phi(x)\phi(y)$. Then

$$\frac{\delta A(\phi)}{\delta\phi(x)} = \int \mathrm{d}y f(x,y)\phi(y) + \int \mathrm{d}y f(y,x)\phi(y),$$
$$\frac{\delta^2 A(\phi)}{\delta\phi(x)\delta\phi(y)} = \frac{\delta}{\delta\phi(y)}\frac{\delta A(\phi)}{\delta\phi(x)} = f(x,y) + f(y,x).$$

More generally, if

$$A(\phi) = \int \mathrm{d}x_1 \ldots \mathrm{d}x_n f(x_1,\ldots,x_n)\phi(x_1)\ldots\phi(x_n),$$

then

$$\frac{\delta^n A(\phi)}{\delta\phi(x_1)\cdots\delta\phi(x_n)} = \sum_{\sigma\in\mathcal{S}_n} f(x_{\sigma(1)},\ldots,x_{\sigma(n)}),$$

where $\mathcal{S}_n$ is the set of permutations of n elements.

In practice, $A(\phi)$ is often a Green function. Take the Green function defined by $(\Delta_x - \phi(x))A(\phi;x,y) = \delta(x-y)$, which can be written $A(\phi) = (\Delta-\phi)^{-1}$. To calculate the functional derivative, we put $A(\phi+\epsilon\psi) = (\Delta-\phi-\epsilon\psi)^{-1}$. The operator identity $Y^{-1} = X^{-1} + X^{-1}(X-Y)Y^{-1}$ gives us $A(\phi+\epsilon\psi) = A(\phi) + \epsilon A(\phi)\psi A(\phi+\epsilon\psi)$. Thus, taking the limit $\epsilon \to 0$,

$$\frac{\delta A(\phi;x,y)}{\delta\psi} = \int \mathrm{d}s A(\phi;x,s)\psi(s)A(\phi;s,y).$$

If we choose now the distribution $\psi(s) = \delta(s-z)$ we find

$$\frac{\delta A(\phi;x,y)}{\delta\phi(z)} = A(\phi;x,z)A(\phi;z,y). \tag{2}$$

This identity will be used repeatedly in the sequel.

3.2 A simple Schwinger equation

As an introduction to the method of planar binary trees we consider the Schwinger equation

$$X = A + F(X, \frac{\delta X}{\delta v(z)}), \tag{3}$$

where A is a functional of v, F is linear in X and $\delta X/\delta v(z)$, and z is a variable over which F integrates. In an equation like (3), F is called the integral operator of the equation and A is called its initial data.

A common example of such an equation is obtained when $X = X(x,y)$, the initial data $A(x,y)$ are the Green function $(\Delta_x - v(x))A(x,y) = \delta(x-y)$ discussed in Sect. 3.1 and the integral operator is

$$F(X, \frac{\delta X}{\delta v(z)})(x,y) = \int \mathrm{d}s\mathrm{d}z X(x,s)f(s,z)\frac{\delta X(s,y)}{\delta v(z)},$$

for some function $f(s,z)$.

Such a Schwinger equation summarizes an infinity of equations that can be obtained by taking successive functional derivatives of (3) with respect to $v(z)$.

$$\begin{aligned} X &= A + F(X, \frac{\delta X}{\delta v(z)}), \\ \frac{\delta X}{\delta v(z_1)} &= \frac{\delta A}{\delta v(z_1)} + F(\frac{\delta X}{\delta v(z_1)}, \frac{\delta X}{\delta v(z)}) \\ &\quad + F(X, \frac{\delta^2 X}{\delta v(z_1)\delta v(z)}), \\ &\ldots \end{aligned}$$

When we take the nth functional derivative of both sides of the $(n-1)$th equation with respect to $v(z_n)$, the equation gets an additional variable z_n, and the chain rule is used to apply $\delta/\delta v(z_n)$ to the right-hand side of the $(n-1)$th equation.

If this is iterated to all values of n, we obtain an infinite system of non-linear integral equations. This system seems difficult to solve because the nth differential of X depends on the kth differentials of X for $k = 0$ to $n+1$.

3.3 The series solution

To write the solution of (3), we must introduce some notation. Sets of arguments will often be needed, so we write $\{z\}_{i,j} = z_i, z_{i+1}, \ldots, z_j$, ($\{z\}_{i,j} = \varnothing$ if $j < i$). Furthermore, if $f(\{z\}_{1,n}) = f(z_1,\ldots,z_n)$ is a function of n variables, then $f_\Sigma(\{z\}_{1,n})$ is defined as the sum of n terms, where the first variable z_1 is shifted step by step from the first to the nth position:

$$f_\Sigma(\{z\}_{1,n}) = f(\{z\}_{1,n}) + f(z_2, z_1, \{z\}_{3,n}) + \cdots \\ + f(\{z\}_{2,n-1}, z_1, z_n) + f(\{z\}_{2,n}, z_1).$$

For each planar binary tree t, we define an infinite dimensional vector $\phi(t)$, with components $\phi^n(t)$, where n goes from 0 to infinity. The nth component is a function of n variables $z_1, \ldots, z_n$. Now we define the initial data. For $t = \bullet$, we take $\phi^0(\bullet) = A$ and

$$\phi^1(\bullet; z_1) = \frac{\delta A}{\delta v(z_1)}. \quad (4)$$

The most natural choice is to define $\phi^n(\bullet)$ as $1/n!$ times the nth functional derivative of A with respect to $v(z)$, but this is not always the most economical choice in practice. The only condition that we need for $n > 1$ is

$$\frac{\delta \phi^{n-1}(\bullet; \{z\}_{1,n-1})}{\delta v(z_n)} = \phi^n_\Sigma(\bullet; \{z\}_{1,n}). \quad (5)$$

When $A(x, y)$ is a Green function of the kind discussed in Sect. 3.1, it is not difficult to build such a $\phi(\bullet)$ from the given initial data A. We can use (2) to show that

$$\begin{aligned} \phi^0(\bullet) &= A(x, y), \\ \phi^1(\bullet; z_1) &= A(x, z_1)A(z_1, y), \\ &\ldots \\ \phi^n(\bullet; \{z\}_{1,n}) &= A(x, z_1)A(z_1, z_2)\ldots A(z_n, y), \end{aligned} \quad (6)$$

satisfy (4) and condition (5).

With this notation we can now write the solution of (3) as

$$X = \sum_t \phi^0(t), \quad (7)$$

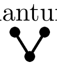
where t spans the set of planar binary trees. Moreover, we have

$$\begin{aligned} \frac{\delta X}{\delta v(z_1)} &= \sum_t \phi^1(t; z_1), \\ &\ldots \\ \frac{\delta^n X}{\delta v(z_1)\cdots\delta v(z_n)} &= \sum_{\sigma\in\mathcal{S}_n}\sum_t \phi^n(t; z_{\sigma(1)}, \ldots, z_{\sigma(n)}). \end{aligned}$$

For each planar binary tree t, the vector $\phi(t)$ is calculated as a function of the vectors $\phi(t_1)$ and $\phi(t_2)$, where t_1 and t_2 are the branches of t. Since $\phi(\bullet)$ is defined from (4), this defines $\phi(t)$ recursively. The recursive definition of $\phi(t)$ is given explicitly by

$$\phi^n(t; \{z\}_{1,n}) = \sum_{k=0}^n F\big(\phi^k(t_1; \{z\}_{1,k}), \\ \phi^{n-k+1}_\Sigma(t_2; z, \{z\}_{k+1,n})\big), \quad (8)$$

for $t \neq \bullet$ and $\phi^n(\bullet)$ is defined in (6).

In a quantum field interpretation, $\bullet$ represents the bare fields,  represents the interaction, and the sum over trees represents all the combinations of the interaction that give the full propagator.

A proof of (7) and (8) is given in the Appendix.

3.4 Enumeration

If the initial data A are such that $\phi^n(\bullet)$ has only one term, as in (6), the chain rule applied to the functional derivative gives a number of terms for $\phi^n(t)$ that we denote $|\phi^n(t)|$. Equation (8) gives us the following recurrence relation for $|\phi^n(t)|$:

$$\begin{aligned} |\phi^n(t)| &= \sum_{k=0}^n (n-k+1)|\phi^k(t_1)||\phi^{n-k+1}(t_2)|, \\ |\phi^n(\bullet)| &= 1. \end{aligned}$$

Using the binomial identity

$$\sum_{k=0}^n \binom{a+k}{a}\binom{b+n-k}{b} = \binom{a+b+n+1}{a+b+1},$$

it can be shown that the solution of this equation is

$$|\phi^n(t)| = \bar\varphi(t)\binom{2|t|+n}{2|t|},$$

where $\bar\varphi(t)$ is an integer which depends only on the tree t (not on n) and is defined recursively by

$$\begin{aligned} \bar\varphi(t) &= \bar\varphi(t_1)(2|t_2|+1)\bar\varphi(t_2), \\ \bar\varphi(\bullet) &= 1. \end{aligned}$$

t_1 and t_2 are the two branches of t.

3.5 A compact notation

To write a compact expression for the recursive definition of $\phi(t)$ we define the deconcatenation of $(z_1, \ldots, z_n)$ by [13]

$$\Delta(z_1, \ldots, z_n) = \sum_{i=0}^n (z_1, \ldots, z_i) \otimes (z_{i+1}, \ldots, z_n).$$

If z belongs to the vector space V, the map $\phi(t)$ acts on the tensor module (Fock space) $T(V) = \bigoplus_{n=0}^\infty V^{\otimes n}$. Sometimes, as in QED, ϕ is defined on $T(V) \times M$, where M is a fixed vector space, for instance $M = V^2$ for the photon propagator.

We define the operator $d(z)$ by

$$d(z)\phi^n(t; z_1, \ldots, z_n) = \phi^{n+1}_\Sigma(t; z, z_1, \ldots, z_n).$$

The recursive definition of ϕ becomes

$$\phi(t) = F \circ \big(Id \otimes d(z)\big) \circ (\phi(t_1) \otimes \phi(t_2)) \circ \Delta,$$

where $F(a \otimes b) = F(a, b)$.

It would be interesting to find a family of equations such that, for any ϕ from $T(V)$ to $\mathbb{C}$, there is a member of the family of which ϕ is a solution. This would generalize Butcher's density theorem[14], and would provide a general class of equations that would be satisfied by the renormalized Green functions.

3.6 Algebra structure

In the case of rooted trees, Butcher [15] has defined a group structure of Runge–Kutta methods that Hairer and Wanner [16] interpreted as a composition of Butcher series. A similar approach can be used for planar binary trees. A powerful aspect of Butcher's approach is that algebraic operations are defined on two spaces at the same time: the space of Runge–Kutta methods, and the space of maps over trees. The same strategy will be used here, and the operations will be defined on the space of integral operators and on the space of maps over planar binary trees.

We start with the addition. If we have two Schwinger equations $X = A + F(X, \delta X/\delta v)$ and $Y = B + G(Y, \delta Y/\delta v)$, the addition of the integral operators is $H = F + G$ and the addition of the maps ϕ (corresponding to the first equation) and ψ (for the second equation) is defined recursively by $\chi^n(\bullet) = \phi^n(\bullet) + \psi^n(\bullet)$, for the initial data $A + B$ and

$$\begin{aligned}&\chi^n(t;\{z\}_{1,n})\\&=\sum_{k=0}^{n} F\big(\chi^k(t_1;\{z\}_{1,k}),\chi_\Sigma^{n-k+1}(t_2;z,\{z\}_{k+1,n})\big)\\&\quad+\sum_{k=0}^{n} G\big(\chi^k(t_1;\{z\}_{1,k}),\chi_\Sigma^{n-k+1}(t_2;z,\{z\}_{k+1,n})\big).\end{aligned}$$

This addition defines clearly a commutative group structure for the integral operators. It also defines a commutative group structure for the space of maps, where the unit element is $\phi(t) = 0$ for all t, and the opposite of $\phi(t)$ is $\psi(t) = -(-1)^{|t|}\phi(t)$.

Multiplication by a scalar λ is similarly defined. An integral operator F becomes λF, and the corresponding map $\phi(t)$ becomes $\lambda^{|t|}\phi(t)$. Notice that maps are not equivalent to integral operators since they contain the initial data too. The present definition of the multiplication by a scalar corresponds to the case where the initial data are not changed. If the initial data are also multiplied by λ, then $\phi(t)$ becomes $\lambda^{|t|+1}\phi(t)$.

This addition is useful when a Schwinger equation is the sum of various terms. Now we can proceed and define another operation coming from a composition of solutions. If we start from two Schwinger equations $X = A + F(X, \delta X/\delta v)$ and $Y = B + G(Y, \delta Y/\delta v)$, the composition of the solutions is defined as the Y obtained with the initial data $B = X$.

It is shown in the Appendix that if χ is the map corresponding to Y (i.e. $Y = \sum_t \chi(t)$), then $\chi(t) = \phi(t) + \psi(t)$, where $\phi(t)$ is the map associated with the equation for X and $\psi^n(t)$ is given by $\psi^n(\bullet) = 0$ and

$$\begin{aligned}\psi^n(t;\{z\}_{1,n}) &= \sum_{k=0}^{n} G\big(\phi^k(t_1;\{z\}_{1,k})+\psi^k(t_1;\{z\}_{1,k}),\\&\quad\phi_\Sigma^{n-k+1}(t_2;z,\{z\}_{k+1,n})+\psi_\Sigma^{n-k+1}(t_2;z,\{z\}_{k+1,n})\big). \quad (9)\end{aligned}$$

This defines a product of integral operators and of maps. In the first proof of (9) given in the Appendix, the integral operator corresponding to this product is constructed. Notice that this operator acts on vectors $\binom{X}{X+Y}$. This product has a unit element (given by $G = 0$).

In the next section, the present method will be applied to the example of QED.

4 The case of QED

We work in the flat Minkowski space with a diagonal metric g (the diagonal is $(1, -1, -1, -1)$). The electron charge is $e = -|e|$. Repeated indices are summed over.

In 1951, Schwinger [6] devised coupled equations involving functional derivatives of $S(x, y; J)$, the full fermion propagator of QED in the presence of an external electromagnetic source $J_\mu(x)$:

$$\begin{aligned}[\Box g_{\mu\nu} - (1-\xi)\partial_\mu\partial_\nu]A^\nu(x;J) &= -J_\mu(x)\\&\quad -\mathrm{i}e\,\mathrm{tr}[\gamma_\mu S(x,x;J)],\end{aligned}$$

$$\begin{aligned}&\Big[\mathrm{i}\gamma^\mu\partial_\mu - m - e\gamma^\mu A_\mu(x;J) + \mathrm{i}e\gamma^\mu\frac{\delta}{\delta J_\mu(x)}\Big]S(x,y;J)\\&= \delta(x-y). \quad (10)\end{aligned}$$

Building on a work by Polivanov [17], Bogoliubov and Shirkov [18] transformed this equation into a Schwinger equation coupling the full fermion propagator $S(x, y)$ with the full photon propagator $D_{\mu\nu}(x, y)$:

$$\begin{aligned}[\Box g^{\mu\nu} - (1-\xi)\partial^\mu\partial^\nu]D_{\nu\rho}(x,y) &= g^\mu{}_\rho\delta(x-y)\\&\quad -\mathrm{i}e\int \mathrm{d}^4z\,\mathrm{tr}\Big[\gamma^\mu\frac{\delta S(x,x;A)}{\delta A_\nu(z)}\Big]D_{\nu\rho}(z,y;A), \quad (11)\\ \big[\mathrm{i}\gamma^\mu\partial_\mu - m - e\gamma^\mu A_\mu(x)\big]S(x,y;A) &= \delta(x-y)\\&\quad +\mathrm{i}e\int \mathrm{d}^4z\,\gamma^\mu D_{\mu\rho}(x,z;A)\frac{\delta S(x,y;A)}{\delta A_\rho(z)}, \quad (12)\end{aligned}$$

where $A(x)$ is now an external electromagnetic field. As explained in [18], (11) and (12) are not completely equivalent to (10), they are valid in the limit $A = 0$ ($J = 0$), which is the standard case of QED.

Multiplying (11) by the bare photon propagator, $D^0_{\mu\nu}(x, y)$ and (12) by the bare fermion propagator in the presence of A, $S^0(x, y; A) = [\mathrm{i}\gamma^\mu\partial_\mu - m - e\gamma^\mu A_\mu]^{-1}$, we obtain our starting Schwinger equations:

$$\begin{aligned}D_{\mu\nu}(x,y;A) &= D^0_{\mu\nu}(x,y) - \mathrm{i}e\int \mathrm{d}^4z\mathrm{d}^4z' D^0_{\mu\lambda}(x,z)\\&\quad\times\mathrm{tr}\Big[\gamma^\lambda\frac{\delta S(z,z;A)}{\delta A_{\lambda'}(z')}\Big]D_{\lambda'\nu}(z',y;A),\\ S(x,y;A) &= S^0(x,y;A) + \mathrm{i}e\int \mathrm{d}^4z\mathrm{d}^4z' S^0(x,z;A)\\&\quad\times\gamma^\lambda D_{\lambda\lambda'}(z,z';A)\frac{\delta S(z,y;A)}{\delta A_{\lambda'}(z')}. \quad (13)\end{aligned}$$

In principle, these equations fully determine $S(x, y; A)$ and $D_{\mu\nu}(x, y; A)$.

4.1 The tree solution

The method of the previous section is now used to write the solution of (13). Since $S^0(x,z;A)$ depends on the external potential $A(x)$, the small extension presented in the Appendix is required. All quantities will be taken at $A(x)=0$, so the external potential will not be mentioned (e.g. $S(x,y)$ means $S(x,y;A)$ at $A=0$).

The notation $\{\lambda,z\}_{i,j}=\lambda_i,z_i,\lambda_{i+1},z_{i+1},\ldots,\lambda_j,z_j$ enables us to write the solution as

$$S(x,y)=\sum_t e^{2|t|}\phi^0(t^\bullet;x,y), \qquad (14)$$

$$\frac{\delta S(x,y)}{\delta A_{\lambda_1}(z_1)}=\sum_t e^{2|t|+1}\phi^1(t^\bullet;x,y;\lambda_1,z_1),$$

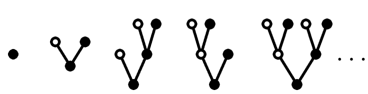
$$D_{\mu\nu}(x,y)=\sum_t e^{2|t|}\phi^0_{\mu\nu}(t^\circ;x,y), \qquad (15)$$

$$\frac{\delta^n S(x,y)}{\delta A_{\lambda_1}(z_1)\cdots\delta A_{\lambda_n}(z_n)}=\sum_{\sigma\in\mathcal{S}_n}\sum_t e^{2|t|+n}\times\phi^n(t^\bullet;x,y;\{\lambda,z\}_{\sigma(1),\sigma(n)}),$$

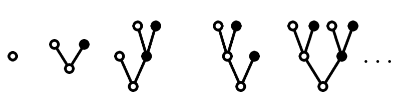
$$\frac{\delta^n D_{\mu\nu}(x,y)}{\delta A_{\lambda_1}(z_1)\cdots\delta A_{\lambda_n}(z_n)}=\sum_{\sigma\in\mathcal{S}_n}\sum_t e^{2|t|+n}\times\phi^n_{\mu\nu}(t^\circ;x,y;\{\lambda,z\}_{\sigma(1),\sigma(n)}).$$

Another change in the notation is that trees now have two colors. The map $\phi(t)$ has a supersymmetric flavor because it has a fermion component for the full fermion propagator and a photon component for the full photon propagator. It is convenient to transfer the fermion/photon index from ϕ to the tree t. Thus, $\phi(t^\circ)$ is the photon component of $\phi(t)$ and $\phi(t^\bullet)$ is its fermion component. Furthermore, $\phi(t^\circ)$ will be drawn with a white root and $\phi(t^\bullet)$ with a black root. All trees are built as $t_1^\circ\vee t_2^\bullet$, where the white (photon) branch is on the left and the black (fermion) branch is on the right.

For example, the fermion trees are



and the photon trees are



Frabetti has calculated the number of trees with $2p+2q+1$ vertices, $p+1$ white leaves and $q+1$ black leaves as [11]

$$c_{p,q}=\frac{(p+q)!}{p!q!}\frac{(p+q+1)!}{(p+1)!(q+1)!}.$$

The following recursive definition gives $\phi(t)$ in terms of $\phi(t_1^\circ)$ and $\phi(t_2^\bullet)$, where t_1° and $t_2^\bullet$ are the branches of t:

$$\begin{aligned}\phi^n(t^\bullet;x,y;\{\lambda,z\}_{1,n})&=\mathrm{i}\sum_{k=0}^{n}\sum_{k'=0}^{n-k}\int \mathrm{d}^4z\mathrm{d}^4z'\\&\times\phi^{n-k-k'}(\bullet;x,z;\{\lambda,z\}_{1,n-k-k'})\\&\times\gamma^\lambda\phi^k_{\lambda\lambda'}(t_1^\circ;z,z';\{\lambda,z\}_{n-k-k'+1,n-k'})\\&\times\phi^{k'+1}_{\Sigma}(t_2^\bullet;z,y;\lambda',z',\{\lambda,z\}_{n-k'+1,n}),\end{aligned} \qquad (16)$$

$$\begin{aligned}\phi^n_{\mu\nu}(t^\circ;x,y;\{\lambda,z\}_{1,n})&=-\mathrm{i}\sum_{k=0}^{n}\int \mathrm{d}^4z\mathrm{d}^4z'D^0_{\mu\lambda}(x,z)\\&\times\mathrm{tr}\big[\gamma^\lambda\phi^{k+1}_{\Sigma}(t_2^\bullet;z,z;\lambda'z',\{\lambda,z\}_{1,k})\big]\\&\times\phi^k_{\lambda'\nu}(t_1^\circ;z',y;\{\lambda,z\}_{k+1,n}).\end{aligned} \qquad (17)$$

This recursive definition is completed by giving the components of $\phi^n(\bullet)$ and $\phi^n(\circ)$:

$$\begin{aligned}\phi^0_{\mu\nu}(\circ;x,y)&=D^0_{\mu\nu}(x,y),\\ \phi^n_{\mu\nu}(\circ;x,y)&=0\quad\text{for}\quad n\ge 1,\\ \phi^0(\bullet;x,y)&=S^0(x,y),\\ \phi^1(\bullet;x,y;\lambda_1,z_1)&=S^0(x,z_1)\gamma^{\lambda_1}S^0(z_1,y),\\ \phi^n(\bullet;x,y;\{\lambda,z\}_{1,n})&=S^0(x,z_1)\gamma^{\lambda_1}S^0(z_1,z_2)\gamma^{\lambda_2}\cdots\\&\quad\times\gamma^{\lambda_n}S^0(z_n,y).\end{aligned}$$

In practice, we use the relation

$$\phi^n(\bullet;x,y;\{\lambda,z\}_{1,n})=S^0(x,z_1)\gamma^{\lambda_1}\times\phi^{n-1}(\bullet;z_1,y;\{\lambda,z\}_{2,n})$$

to show that the double sum of (16) can be replaced by

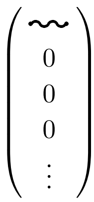
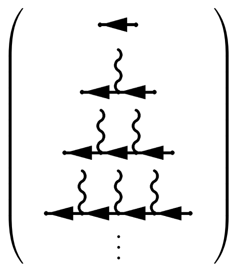
$$\begin{aligned}&\phi^n(t^\bullet;x,y;\{\lambda,z\}_{1,n})\\&=S^0(x,z_1)\gamma^{\lambda_1}\phi^{n-1}(t^\bullet;z_1,y;\{\lambda,z\}_{2,n})\\&+\mathrm{i}\sum_{k=0}^{n}\int\mathrm{d}^4z\mathrm{d}^4z'S^0(x,z)\gamma^\lambda\phi^k_{\lambda\lambda'}(t_1^\circ;z,z';\{\lambda,z\}_{1,k})\\&\times\phi^{n-k+1}_{\Sigma}(t_2^\bullet;z,y;\lambda'z',\{\lambda,z\}_{k+1,n}).\end{aligned} \qquad (18)$$

This is still a recursive definition, but it now uses the smaller component of ϕ for the same tree $t^\bullet$. Notice that the first term of (18) is absent if $n=0$.

Three remarks can be useful at this point. Firstly, considering all quantities at $A=0$, we find as in Sect. 3.1 that $\delta S^0(x,y)/\delta A_\lambda(z)=eS^0(x,z)\gamma^\lambda S^0(z,y)$. Therefore, in the definition of $\phi^n(\bullet)$, a factor e^n was suppressed and transferred to the solution (14) and (15). It must be checked that this is compatible with renormalization. Secondly, it will be shown in Sect. 5.1 that, from Furry's theorem, the components $\phi^n(t^\circ)$ are zero when n is odd. This reduces the sums in (16), (17) and (18) to the even components of $\phi(t_1^\circ)$. Finally, the vector space V of Sect. 3.5 has now become the space $\{0,1,2,3\}\times\mathbb{R}^4$.

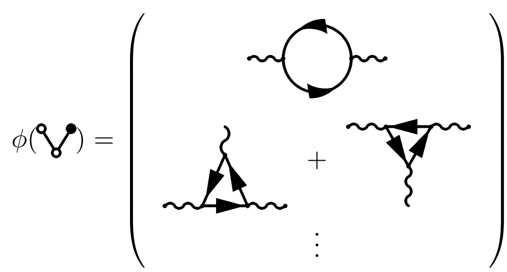
4.2 Diagrammatic interpretation

The recursive solution of the previous sections can be illustrated in the usual diagrammatic language. If we put point x on the left and point y on the right, we can draw

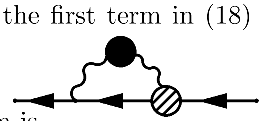
$\phi(\circ) =$  $\phi(\bullet) =$ 

In other words, $\phi^n(\bullet)$ is a propagator with n dangling photon lines (n starts at 0). More generally, all the components $\phi^n(t^\circ)$ or $\phi^n(t^\bullet)$ have n dangling photon lines. To see the action of the recursive equation for the photon, observe that, in (17) the fermion extremities of $t_2^\bullet$ are closed on an additional photon line by a bare vertex (on the left), and each dangling photon line of $t_2^\bullet$ is linked in turn to the photon extremities of t_1° (on the right). Diagrammatically:

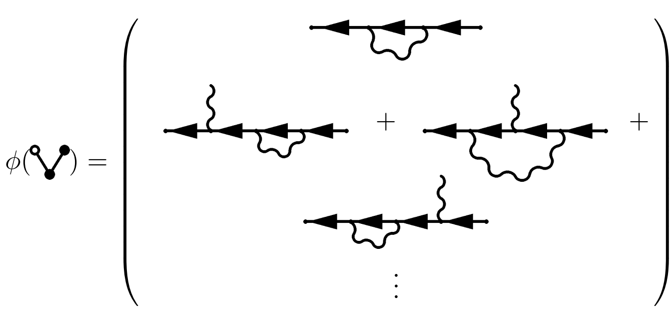
The next term is



For the fermion propagator, in (16) and (18), each dangling photon line of $t_2^\bullet$ is linked in turn to the right photon extremity of t_1°. The left photon extremity of t_1° is linked to the left extremity of the electron propagator $t_2^\bullet$ by an additional bare vertex.

If we neglect the first term in (18) we can write diagrammatically: 

The next term is



In all these diagrams for $S(x,y)$ and $D_{\lambda\mu}(x,y)$, x is on the left and y on the right. Notice that each component of $\phi(t)$ has $|t|$ loops.

4.3 Diagram enumeration

As a warm-up exercise, we can calculate the number of diagrams in the component $\phi^n(t)$, that we denote $|\phi^n(t)|$. From (16) and (17) we find the equations for $|\phi^n(t^\circ)|$ and $|\phi^n(t^\bullet)|$

$$|\phi^n(t^\bullet)| = \sum_{k=0}^{n}\sum_{k'=0}^{n-k}(k'+1)|\phi^k(t_1^\circ)||\phi^{k'+1}(t_2^\bullet)|,$$
$$|\phi^n(t^\circ)| = \sum_{k=0}^{n}(n-k+1)|\phi^k(t_1^\circ)||\phi^{n-k+1}(t_2^\bullet)|,$$

with $\phi^n(\bullet) = 1$, $\phi^n(\circ) = \delta_{n,0}$.

The solution of this recursive equation is

$$|\phi^n(t^\bullet)| = \bar\varphi(t)\binom{2|t|+n}{n},$$
$$|\phi^n(t^\circ)| = \bar\varphi(t)\binom{2|t|+n-1}{n},$$

where $\bar\varphi(t)$ does not depend on the colour of t and was defined in Sect. 3.4.

This can be used to calculate the total number of diagrams for the electron or photon propagators (the number is the same) contributing at e^{2n}. Let us define

$$s_n = \sum_{t\in Y_n}\bar\varphi(t) = \sum_{t\in Y_n}\bar\varphi(t_1)(2|t_2|+1)\bar\varphi(t_2).$$

Using (1) for $n>0$ we find

$$s_n = \sum_{k=0}^{n-1}\sum_{|t_1|=k}\bar\varphi(t_1)\sum_{|t_2|=n-k-1}(2|t_2|+1)\bar\varphi(t_2)$$
$$= \sum_{k=0}^{n-1}(2k+1)s_k s_{n-k-1}.$$

The starting value is $s_0 = 1$. For $n = 0,1,2,3,4,5$ this gives us $1,1,4,27,248,2830$, in agreement with [19,20]. The generating function $y(x)$ for the sequence s_n satisfies the differential equation $2x^2yy' + xy^2 - y + 1 = 0$ with $y(0) = 1$.

This enumation takes into account neither symmetry nor Furry's theorem, which says that $|\phi^n(t^\circ)| = 0$ if n is odd. The main point of this enumeration is to show that each tree represents the sum of a large number of diagrams when $|t|$ is large. This may prove useful for practical calculations.

4.4 Fourier transform

In applications, it is often convenient to work in the k space. The Fourier transform of $\phi(t;x,y;\{\lambda,z\}_{1,n})$ is defined by

$$\psi(t;q,q';\{\lambda,p\}_{1,n}) = \int \mathrm{d}^4x\mathrm{d}^4y\mathrm{d}^4z_1\ldots\mathrm{d}^4z_n$$
$$\times \mathrm{e}^{\mathrm{i}(q\cdot x - q'\cdot y + p_1\cdot z_1 + \cdots + p_n\cdot z_n)}\phi(t;x,y;\{\lambda,z\}_{1,n}).$$

This corresponds to outgoing momenta p_i along the dangling photon lines. If this is introduced into (16) and (17), we find

$$\psi(t;q,q';\{\lambda,p\}_{1,n}) = (2\pi)^4\delta(q+p_1+\cdots+p_n-q') \times\tilde{\phi}(t;q;\{\lambda,p\}_{1,n}).$$

The full fermion and photon propagators in Fourier space are

$$S(q) = \sum_t e^{2|t|}\tilde{\phi}^0(t^\bullet;q),$$
$$D_{\lambda\mu}(q) = \sum_t e^{2|t|}\tilde{\phi}^0_{\lambda\mu}(t^\circ;q).$$

Here $\tilde{\phi}(t)$ satisfies the recursive relation

$$\begin{aligned}\tilde{\phi}^n(t^\bullet;q;\{\lambda,p\}_{1,n}) &= S^0(q)\gamma^{\lambda_1} \times\tilde{\phi}^{n-1}(t^\bullet;q+p_1;\{\lambda,p\}_{2,n})\\ &+\mathrm{i}\sum_{k=0}^{n}\int\frac{\mathrm{d}^4p}{(2\pi)^4}S^0(q)\gamma^\lambda\tilde{\phi}^k_{\lambda\lambda'}(t_1^\circ;p;\{\lambda,p\}_{1,k})\\ &\times\tilde{\phi}^{n-k+1}_{\Sigma}(t_2^\bullet;q-p;\lambda',p+P_k,\{\lambda,p\}_{k+1,n}), \quad (19)\end{aligned}$$

$$\begin{aligned}\tilde{\phi}^n_{\mu\nu}(t^\circ;q;\{\lambda,p\}_{1,n}) &= -\mathrm{i}\sum_{k=0}^{n}\int\frac{\mathrm{d}^4p}{(2\pi)^4}D^0_{\mu\lambda}(q)\\ &\times\mathrm{tr}\big[\gamma^\lambda\tilde{\phi}^{k+1}_{\Sigma}(t_2^\bullet;p;\lambda',-q-P_k,\{\lambda,p\}_{1,k})\big]\\ &\times\tilde{\phi}^{n-k}_{\lambda'\nu}(t_1^\circ;q+P_k;\{\lambda,p\}_{k+1,n}), \quad (20)\end{aligned}$$

where we have noted $P_k = p_1+\cdots+p_k$, $(P_0 = 0)$ and with the initial data

$$\begin{aligned}\tilde{\phi}^0_{\mu\nu}(\circ;q) &= D^0_{\mu\nu}(q)\\ &= -\frac{g_{\mu\nu}}{q^2+\mathrm{i}\epsilon}+(1-1/\xi)\frac{q_\mu q_\nu}{(q^2+i\epsilon)^2},\\ \tilde{\phi}^n_{\mu\nu}(\circ;q) &= 0 \quad\text{for}\quad n\geq 1,\\ \tilde{\phi}^0(\bullet;q) &= S^0(q) = (\gamma^\mu q_\mu - m+\mathrm{i}\epsilon)^{-1},\\ \tilde{\phi}^1(\bullet;q;\lambda_1,p_1) &= S^0(q)\gamma^{\lambda_1}S^0(q+p_1),\\ \tilde{\phi}^n(\bullet;q;\{\lambda,p\}_{1,n}) &= S^0(q)\gamma^{\lambda_1}S^0(q+p_1)\gamma^{\lambda_2}\cdots\gamma^{\lambda_n}\\ &\times S^0(q+p_1+\cdots+p_n).\end{aligned}$$

As for the real space case, the first term of (19) is absent for $n=0$ and

$$\begin{aligned}\tilde{\phi}^n_{\Sigma}(t;q;\{\lambda,p\}_{1,n}) &= \tilde{\phi}^n(t;q;\{\lambda,p\}_{1,n})\\ &+\tilde{\phi}^n(t;q;\lambda_2,p_2,\lambda_1,p_1,\{\lambda,p\}_{3,n})+\cdots\\ &+\tilde{\phi}^n(t;q;\{\lambda,p\}_{2,n},\lambda_1,p_1).\end{aligned}$$

Again, Furry's theorem enables us to restrict the sum to the even components of $\tilde{\phi}_{\mu\nu}(t^\circ)$.

Finally, in (20), t_1° intervenes only as a factor. Thus, we can factorize (20) as

$$\begin{aligned}\tilde{\phi}^n_{\mu\nu}(t^\circ;q;\{\lambda,p\}_{1,n}) &= \sum_{k=0}^{n}\tilde{\phi}^k_{\mu\lambda}(\circ\vee t_2^\bullet;q;\{\lambda,p\}_{1,k})\\ &\times[(D^0)^{-1}]^{\lambda\lambda'}(q+P_k)\tilde{\phi}^{n-k}_{\lambda'\nu}(t_1^\circ;q+P_k;\{\lambda,p\}_{k+1,n}).\end{aligned}$$

5 Applications

In this section, the previous results are applied to Furry's theorem and the Ward–Takahashi identities. Finally, a sum over trees is defined for the two-particle Green function.

5.1 Furry's theorem

Within our approach, Furry's theorem implies $\phi^n(t^\circ) = 0$ for odd n. To show this, remark that, in (20), the integral

$$\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\lambda\tilde{\phi}^{n-k+1}_{\Sigma}(t_2^\bullet;p;\lambda',-q-P_k,\{\lambda,p\}_{k+1,n})\big]$$

is a fermion loop with $n-k+2$ external photon lines. According to Furry's theorem, this loop is zero when $n-k$ is odd. From its explicit definition, $\phi^n(\circ) = 0$ is zero if n is odd (in fact, it is zero if $n\geq 1$). We reason recursively on the number of vertices of t°. If $\phi^n(t^\circ) = 0$ for odd n and t° with up to $2N-1$ vertices, take t° with $2N+1$ vertices. In (20), the integral over p is zero if $n-k$ is odd (Furry's theorem) and $\phi^k(t_1^\circ)$ is zero if k is odd (because t_1° has less vertices than t). Thus, for the left-hand side of (20) to be eventually non-zero, k and $n-k$ must be even, so n must be even. Therefore, $\phi^n(t^\circ) = 0$ for odd n.

5.2 Ward–Takahashi identities

Within the present approach, the Ward–Takahashi identities, as formulated in [21] or [22], take the form

$$\sum_\mu k_\mu\tilde{\phi}^{n+1}_{\Sigma}(t^\bullet;q;\mu,k,\{\lambda,p\}_{1,n}) = \tilde{\phi}^n(t^\bullet;q;\{\lambda,p\}_{1,n}) -\tilde{\phi}^n(t^\bullet;q+k;\{\lambda,p\}_{1,n}). \quad (21)$$

In other words, the trees are compatible with the Ward–Takahashi identities, which induce a boundary operator on $\phi(t)$. To prove this, we need to start from the fully symmetrized form of $\tilde{\phi}$:

$$\tilde{\phi}^n_{\Sigma^n}(t;q;\{\lambda,p\}_{1,n}) = \sum_{\sigma\in\mathcal{S}_n}\tilde{\phi}^n(t;q;\{\lambda,p\}_{\sigma(1),\sigma(n)}).$$

We follow the notation of [11]. Let E_n be the set of $\tilde{\phi}^n_{\Sigma^n}(t;q;\{\lambda,p\}_{1,n})$ obtained by varying the gauge parameter ξ. We define the face maps d_i by

$$\begin{aligned}&d_i\tilde{\phi}^n_{\Sigma^n}(t;q;\{\lambda,p\}_{1,i-1},\mu,k,\{\lambda,p\}_{i+1,n}) =\\ &\sum_{\mu=0}^{3}k_\mu\tilde{\phi}^n_{\Sigma^n}(t;q;\{\lambda,p\}_{1,i-1},\mu,k,\{\lambda,p\}_{i+1,n})\\ &= \tilde{\phi}^{n-1}_{\Sigma^{n-1}}(t;q;\{\lambda,p\}_{1,i-1},\{\lambda,p\}_{i+1,n})\\ &-\tilde{\phi}^{n-1}_{\Sigma^{n-1}}(t;q+k;\{\lambda,p\}_{1,i-1},\{\lambda,p\}_{i+1,n}),\end{aligned}$$

where the last line is the Ward–Takahashi identity. The index i of d_i means that d_i acts on the ith argument in the list $\lambda_1, p_1, \ldots, \lambda_n, p_n$. From the definition it can be checked that $d_i d_j = d_{j-1} d_i$ for $i < j$. Therefore, the face maps generate the boundary operator $d = \sum_{i=1}^{n}(-1)^i d_i$ which satisfies $d \circ d = 0$ (see [11]). Hence, $\{E_n, d_i\}$ is a presimplicial set that gives rise to a chain complex $(k[E_*], d)$.

Notice that, for $n = 0$, we have put $d\tilde{\phi}^0(t) = 0$.

5.3 Two-particle Green function

In [6] Schwinger proceeds by giving an equation for the full two-particle fermion Green function:

$$\Big[\mathrm{i}\gamma^\mu \partial_{x_1^\mu} - m - e\gamma^\mu A_\mu(x_1; J) + \mathrm{i}e\gamma^\mu \frac{\delta}{\delta J_\mu(x_1)}\Big] \times S(x_1, x_2; y_1, y_2; J) = \delta(x_1 - y_1) S(x_2, y_2; J) - \delta(x_1 - y_2) S(x_2, y_1; J).$$

From the identity [18]

$$\frac{\delta}{\delta J^\mu(x)} = -\int \mathrm{d}y G_{\lambda\mu}(y, x) \frac{\delta}{\delta A_\lambda(y)},$$

we go from the J to the A variable (in the limit A=0)

$$\begin{aligned}&\big[\mathrm{i}\gamma^\mu \partial_{x_1^\mu} - m - e\gamma^\mu A_\mu(x_1)\big] S(x_1, x_2; y_1, y_2; A)\\ &= \delta(x_1 - y_1) S(x_2, y_2; A) - \delta(x_1 - y_2) S(x_2, y_1; A)\\ &+\mathrm{i}e\int \mathrm{d}^4 z \gamma^\mu \frac{\delta S(x_1, x_2; y_1, y_2; A)}{\delta A_\lambda(z)} G_{\lambda\mu}(z, x_1).\end{aligned}$$

It remains to multiply by S^0 and to take $A = 0$. This yields

$$\begin{aligned}&S(x_1, x_2; y_1, y_2)\\ &= S^0(x_1, y_1) S(x_2, y_2) - S^0(x_1, y_2) S(x_2, y_1) \qquad (22)\\ &+\mathrm{i}e\int \mathrm{d}^4 z \mathrm{d}^4 z' S^0(x_1, z)\gamma^\mu \frac{\delta S(z, x_2; y_1, y_2)}{\delta A_\lambda(z')} G_{\lambda\mu}(z', z).\end{aligned}$$

In (22), $S(x, y)$ plays the role of the initial data, whereas it is the solution of (13). Therefore, we are making a composition of solutions, which was considered in Sect. 3.6. The situation is not exactly the same as that of Sect. 3.6, but the proof is similar (only notationally more cumbersome) and we obtain an expression for the two-particle Green function as a sum over planar binary trees

$$S(x_1, x_2; y_1, y_2) = \sum_t e^{2|t|} \chi^0(t; x_1, x_2; y_1, y_2).$$

According to the rule of composition, χ is the sum of three terms:

$$\begin{aligned}\chi^n(t; x_1, x_2; y_1, y_2; \{\lambda, z\}_{1,n}) &= S^0(x_1, y_1)\phi^n(t^\bullet; x_2, y_2; \{\lambda, z\}_{1,n})\\ &-S^0(x_1, y_2)\phi^n(t^\bullet; x_2, y_1; \{\lambda, z\}_{1,n})\\ &+\psi^n(t; x_1, x_2; y_1, y_2; \{\lambda, z\}_{1,n}),\end{aligned}$$

and ψ itself is given by

$$\begin{aligned}\psi^n(t; x_1, x_2; y_1, y_2; \{\lambda, z\}_{1,n}) &= S^0(x_1, z_1)\gamma^{\lambda_1}\\ &\times\chi^{n-1}(t; z_1, x_2; y_1, y_2; \{\lambda, z\}_{2,n})\\ &+\mathrm{i}\sum_{k=0}^{n}\int \mathrm{d}^4 z \mathrm{d}^4 z' S^0(x_1, z)\gamma^\mu \phi^k_{\lambda\mu}(t_1^\circ; z', z; \{\lambda, z\}_{1,k})\\ &\times\chi_\Sigma^{n-k+1}(t_2; z, x_2; y_1, y_2; \lambda, z', \{\lambda, z\}_{k+1,n}). \qquad (23)\end{aligned}$$

Equation (23) is not very simple, but it provides a way to recursively calculate all orders of the perturbation expansion for the two-particle Green function of QED. In that sense, it is not so complicated.

6 Self-energy and vacuum polarization

To calculate the self-energy, we introduce a further operation on planar binary trees.

6.1 The pruning operator

The pruning operator P applied to tree t is defined by

$$P(t) = \sum_{i=1}^{n(t)} u_i \otimes v_i, \qquad (24)$$

where $n(t)$ is an integer recursively defined by

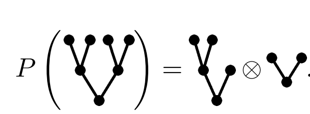
$$\begin{aligned}&n(\bullet) = 0,\\ &n(t) = 0 \quad \text{if} \quad t = t_1 \vee \bullet,\\ &n(t) = 1 + n(t_2) \quad \text{if} \quad t = t_1 \vee t_2, \quad t_2 \neq \bullet,\end{aligned}$$

and the planar binary trees u_i and v_i are determined by

$$\begin{aligned}&P(\bullet) = 0,\\ &P(t) = 0 \quad \text{if} \quad t = t_1 \vee \bullet,\\ &P(t) = (t_1 \vee \bullet) \otimes t_2 + \sum_{i=1}^{n(t_2)} (t_1 \vee u_i) \otimes v_i\\ &\qquad \text{if} \quad t = t_1 \vee t_2, \quad t_2 \neq \bullet. \qquad (25)\end{aligned}$$

The trees u_i and v_i in (25) are generated by (24) for $t = t_2$.

As a more graphical definition, for a tree t, we consider the path starting from the root and climbing up the tree by taking, at each vertex, the right branch. This path terminates at the extreme right leaf of the tree and goes through $n(t)+2$ vertices (including the root and the leaf). For each vertex s_i along that path, excluding the root and the leaf, we cut t in two trees u_i and v_i, where v_i is the subtree of t that has s_i as a root, and u_i the subtree of t that has s_i as a leaf. For example we have



6.2 Products and inversion

From the pruning operator, we can define a convolution of maps over trees [1]. If $\phi(t)$ and $\psi(t)$ are two maps satisfying $\phi(\bullet) = \psi(\bullet) = 0$, we define the convolution of ϕ and ψ by

$$(\phi \star \psi)(t) = \sum_{i=1}^{n(t)} \phi(u_i)\psi(v_i).$$

For infinite dimensional maps, the components are defined analogously. For instance, the nth component of $(\phi\star\psi)(t; x, y)$ is

$$(\phi \star \psi)^n(t; x, y; \{z\}_{1,n}) = \sum_{i=1}^{n(t)}\sum_{k=0}^{n} \int \mathrm{d}s \phi^k(u_i; x, s; \{z\}_{1,k}) \\ \psi^{n-k}(v_i; s, y; \{z\}_{k+1,n}).$$

The convolution is compatible with the product of two series over trees. Starting from two series

$$X(\lambda) = \sum_t \lambda^{|t|}\phi(t),$$
$$Y(\lambda) = \sum_t \lambda^{|t|}\psi(t),$$

where $\phi(\,\bullet\,) = \psi(\,\bullet\,) = 0$, we can use (32) to show that

$$X(\lambda)Y(\lambda) = \sum_t \lambda^{|t|}(\phi \star \psi)(t).$$

Convolution is useful to solve the Schwinger equations of ϕ^3 and ϕ^4 quantum field theories, and to implement renormalization.

For the present paper, we shall use the pruning operator to invert series over trees. We do not use the convolution operation, because we want to specify which kind of trees (with black or white roots) are used in the formulas. This will help calculating the self-energy. If we define Y by

$$X = \sum_t \phi^0(t) = \frac{1}{1/\phi^0(\,\bullet\,) - Y},$$

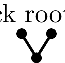
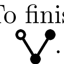
it is proved in the Appendix that $Y = \sum_t \psi^0(t)$, where

$$\psi^0(\,\bullet\,) = 0,$$
$$\psi^0(t) = \frac{1}{\phi^0(\,\bullet\,)}\Big(\phi^0(t)\frac{1}{\phi^0(\,\bullet\,)} - \sum_{i=1}^{n(t)} \phi^0(u_i)\psi^0(v_i)\Big). \quad (26)$$

In this equation, u_i, v_i and $n(t)$ are determined from t by (24), and the sum over i is zero if $P(t) = 0$.

[1] I thank Jean-Louis Loday for drawing my attention to this point.

6.3 Self-energy

To use (26) for the calculation of the self-energy, some precautions are required, because of the presence of black and white vertices. Going through the proof in the Appendix, we see that the proof is still valid when vertices can have two colors with the condition that all the trees considered in Y_n, Y_k and Y_{n-k} of (32) have a black root, and that the grafting operations $t_1 \vee \bullet$ and $t_1 \vee u_i$ make trees with a black root. To finish the proof, it is enough to replace the tree  by .

Therefore, the self-energy is given, in terms of the map $\tilde\phi$ for the full fermion propagator, by

$$\Sigma(q) = \sum_t e^{2|t|}\psi^0(t^\bullet; q),$$

with $\psi^0(\,\bullet\,; q) = 0$ and

$$\psi^0(t^\bullet; q) = (\gamma^\alpha q_\alpha - m)\tilde\phi^0(t^\bullet; q)(\gamma^\beta q_\beta - m) \\ -(\gamma^\alpha q_\alpha - m)\sum_{i=1}^{n(t)} \tilde\phi^0(u_i^\bullet; q)\psi^0(v_i^\bullet; q).$$

6.4 Irreducible vertex

From the formula for the self-energy, we can deduce the complete one-particle irreducible three-point function $\Gamma^\nu(p, p')$; see [23], p. 335. We reintroduce the external potential to write

$$\Sigma(x, y; A) = \sum_t (\mathrm{i}\gamma^\lambda\partial_{x^\lambda} - m - e\gamma^\lambda A_\lambda(x)) \\ \phi(t^\bullet; x, y; A)(-\mathrm{i}\gamma^\mu\overleftarrow{\partial}_{y^\mu} - m - e\gamma^\mu A_\mu(y)) \\ -\sum_t (\mathrm{i}\gamma^\lambda\partial_{x^\lambda} - m - e\gamma^\lambda A_\lambda(x)) \\ \sum_{i=1}^{n(t)} \int \mathrm{d}z' \phi(u_i^\bullet; x, z'; A)\psi(v_i^\bullet; z', y; A),$$

where $\overleftarrow{\partial}_{y^\mu}$ acts on the left. In the real space, $\Gamma^\nu(x, y; z)$ is given

$$\Gamma^\nu(x, y; z) = \frac{\delta\Sigma(x, y; A)}{\delta A_\nu(z)} \quad \text{for} \quad A = 0.$$

Therefore,

$$\Gamma^\nu(x, y; z) = \sum_t e^{2|t|+1}\psi^1(t^\bullet; x, y; \nu, z),$$

with

$$\psi^1(t^\bullet;x,y;\nu,z) = -\gamma^\nu\delta(z-x) \times\phi^0(t^\bullet;x,y)(-\mathrm{i}\gamma^\mu\overleftarrow{\partial}_{y^\mu}-m) -(\mathrm{i}\gamma^\lambda\partial_{x^\lambda}-m)\phi^0(t^\bullet;x,y)\gamma^\nu\delta(z-y) +(\mathrm{i}\gamma^\lambda\partial_{x^\lambda}-m)\phi^1(t^\bullet;x,y;\nu,z)(-\mathrm{i}\gamma^\mu\overleftarrow{\partial}_{y^\mu}-m) +\gamma^\nu\delta(z-x)\sum_{i=1}^{n(t)}\int \mathrm{d}z'\phi^0(u_i^\bullet;x,z')\psi^0(v_i^\bullet;z',y) -(\mathrm{i}\gamma^\lambda\partial_{x^\lambda}-m)\sum_{i=1}^{n(t)}\int \mathrm{d}z'\phi^0(u_i^\bullet;x,z')\psi^1(v_i^\bullet;z',y;\nu,z) -(\mathrm{i}\gamma^\lambda\partial_{x^\lambda}-m)\sum_{i=1}^{n(t)}\int \mathrm{d}z'\phi^1(u_i^\bullet;x,z';\nu,z)\psi^0(v_i^\bullet;z',y).$$

In Fourier space, this gives us

$$\Gamma^\nu(q,q+p) = \sum_t e^{2|t|+1}\psi^1(t^\bullet;q;\nu,p),$$

with

$$\psi^1(t^\bullet;q;\nu,p) = -\gamma^\nu\tilde\phi^0(t^\bullet;q+p)(\gamma^\mu(q_\mu+p_\mu)-m) -(\gamma^\lambda q_\lambda-m)\tilde\phi^0(t^\bullet;q)\gamma^\nu +(\gamma^\lambda q_\lambda-m)\tilde\phi^1(t^\bullet;q;\nu,p)(\gamma^\mu(q_\mu+p_\mu)-m) +\gamma^\nu\sum_{i=1}^{n(t)}\tilde\phi^0(u_i^\bullet;q+p)\psi^0(v_i^\bullet;q+p) -(\gamma^\lambda q_\lambda-m)\sum_{i=1}^{n(t)}\tilde\phi^0(u_i^\bullet;q)\psi^1(v_i^\bullet;q;\nu,p) -(\gamma^\lambda q_\lambda-m)\sum_{i=1}^{n(t)}\tilde\phi^1(u_i^\bullet;q;\nu,p)\psi^0(v_i^\bullet;q+p).$$

6.5 Vacuum polarization

It was observed that the presence of trees with black and white vertices brought about a small complication in the calculation of the electron self-energy. For the vacuum polarization, the change is greater.

We start from an equation writing the full vacuum polarization $\Pi^{\lambda\mu}(q)$ as a fermion loop ([23] p. 477):

$$\Pi^{\lambda\mu}(q) = \mathrm{i}\sum_t e^{2|t|+2}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\lambda\tilde\phi^1(t^\bullet;p;\mu,-q)\big].$$

If this last equation is compared to (17) for $n=0$, we see that they become equal if $t_1^\circ=\circ$ and the free photon propagators are eliminated. This gives us the vacuum polarization in terms of the map ϕ for the full photon propagator, by

$$\Pi_{\lambda\mu}(q) = {\sum_t}' e^{2|t|}\psi^0_{\lambda\mu}(t^\circ;q),$$

where the prime means that the sum is carried out only over the trees $t^\circ=\circ\vee t_2$. $\psi^0_{\lambda\mu}(t^\circ;q)$ is defined by

$$\psi^0_{\lambda\mu}(t^\circ;q) = -(-q^2g_\lambda^{\lambda'}+(1-\xi)q_\lambda q^{\lambda'}) \phi^0_{\lambda'\mu'}(t^\circ;q)(-q^2g^{\mu'}_\mu+(1-\xi)q^{\mu'}q_\mu).$$

If $t^\circ\neq\circ\vee t_2$, $\psi^0_{\lambda\mu}(t^\circ;q)=0$. Moreover, following the discussion of [22], p. 339, the Ward identities imply that the fermion loop in (19) is transverse. Therefore, each $\psi^0_{\lambda\mu}(t^\circ;q)$ is transverse (i.e. $q^\lambda\psi^0_{\lambda\mu}(t^\circ;q)=0$).

7 Interaction with an external field

In this section we come back to the original Schwinger equation, because the presence of an external source is a convenient way to represent the nuclei in the QED of matter.

Starting from (10), we multiply by the corresponding bare Green functions and we introduce $A^\nu(x;J)$ into the second equation to obtain

$$S(x,y;J) = S^0(x,y) - e\int\mathrm{d}^4z\mathrm{d}^4z'S^0(x,z)\gamma^\mu D^0_{\mu\nu}(z,z') \times J^\nu(z')S(z,y;J) - \mathrm{i}e^2\int\mathrm{d}^4z\mathrm{d}^4z'S^0(x,z)\gamma^\mu \times D^0_{\mu\nu}(z,z')\mathrm{tr}\big[\gamma^\nu S(z',z';J)\big]S(z,y;J) -\mathrm{i}e\int\mathrm{d}^4zS^0(x,z)\gamma^\mu\frac{\delta S(z,y;J)}{\delta J^\mu(z)}. \quad (27)$$

Here, the Schwinger equation is a sum of three terms. The first term is simply the classical interaction with the external source $J^\nu(z_2)$; it can be solved by defining a bare propagator in the presence of this source:

$$S^0(x,y;J)^{-1} = \mathrm{i}\gamma^\mu\partial_\mu - m + e\gamma^\mu\int\mathrm{d}^4zD^0_{\mu\nu}(x,z)J^\nu(z).$$

Equation (27) now becomes

$$S(x,y;J) = S^0(x,y;J) - \mathrm{i}e^2\int\mathrm{d}^4z\mathrm{d}^4z'S^0(x,z;J)\gamma^\mu \times D^0_{\mu\nu}(z,z')\mathrm{tr}\big[\gamma^\nu S(z',z';J)\big]S(z,y;J) -\mathrm{i}e\int\mathrm{d}^4zS^0(x,z;J)\gamma^\mu\frac{\delta S(z,y;J)}{\delta J^\mu(z)}. \quad (28)$$

This equation is solved by the usual methods, and the recursive definition of $\phi(t)$ is

$$\phi^n(\bullet;x,y;\{\lambda,z\}_{1,n}) = (-1)^n \int \mathrm{d}^4 s_1 \dots \mathrm{d}^4 s_n S^0(x,s_1;J)$$
$$\times\gamma^{\mu_1} D^0_{\mu_1\lambda_1}(s_1,z_1) S^0(s_1,s_2;J)\dots$$
$$\gamma^{\mu_n} D^0_{\mu_n\lambda_n}(s_n,z_n) S^0(s_n,y;J),$$
$$\phi^n(t;x,y;\{\lambda,z\}_{1,n}) = -\mathrm{i}\sum_{k=0}^{n}\sum_{k'=0}^{n-k}\int \mathrm{d}^4 z \mathrm{d}^4 z'$$
$$\times\phi^{n-k-k'}(\bullet;x,z;\{\lambda,z\}_{1,n-k-k'})\gamma^\mu D^0_{\mu\nu}(z,z')$$
$$\times\mathrm{tr}\big[\gamma^\nu\phi^k(t_1;z',z';\{\lambda,z\}_{n-k-k'+1,n-k'})\big]$$
$$\times\phi^{k'}(t_2;z,y;\{\lambda,z\}_{n-k'+1,n})$$
$$-\mathrm{i}\sum_{k=0}^{n}\int \mathrm{d}^4 z\phi^k(\bullet;x,z;\{\lambda,z\}_{1,k})$$
$$\times\gamma_\mu\phi_\Sigma^{n-k+1}(t_2;z,y;\mu,z,\{\lambda,z\}_{k+1,n}),$$

where the last term is non-zero only if t has the special shape $t = \bullet \vee t_2$.

Here again, the recurrence relation between $\phi^n(\bullet)$ and $\phi^{n-1}(\bullet)$ reduces the number of sums to:

$$\phi^n(t;x,y;\{\lambda,z\}_{1,n}) = \mathrm{i}\int \mathrm{d}^4 s_1 S^0(x,s_1;J)$$
$$\times\gamma^{\mu_1} D^0_{\mu_1\lambda_1}(s_1,z_1)\phi^{n-1}(t;z_1,y;\{\lambda,z\}_{2,n})$$
$$-\mathrm{i}\sum_{k=0}^{n}\int \mathrm{d}^4 z \mathrm{d}^4 z' S^0(x,z;J)\gamma^\mu D^0_{\mu\nu}(z,z')$$
$$\times\mathrm{tr}\big[\gamma^\nu\times\phi^k(t_1;z',z';\{\lambda,z\}_{1,k})\big]\phi^{n-k}(t_2;z,y;\{\lambda,z\}_{k+1,n})$$
$$-\mathrm{i}\int \mathrm{d}^4 z S^0(x,z;J)\gamma_\mu\phi_\Sigma^{n+1}(t_2;z,y;\mu,z,\{\lambda,z\}_{1,n}),$$

where the last term is non-zero only if t has the special shape $t = \bullet \vee t_2$.

According to Schwinger [6], the full photon Green function in the presence of an external current J is given by the functional derivative of $A(x;J)$ with respect to $J(y)$. Therefore

$$D_{\lambda\mu}(x,y;J)$$
$$= -\frac{\delta A_\lambda(x;J)}{\delta J^\mu(y)}$$
$$= D^0_{\lambda\mu}(x,y)$$
$$+\mathrm{i}e\int \mathrm{d}z D^0_{\lambda\nu}(x,z)\mathrm{tr}\big[\gamma^\nu\frac{\delta S(z,z;J)}{\delta J^\mu(y)}\big]$$
$$= D^0_{\lambda\mu}(x,y)$$
$$+\mathrm{i}e^2\sum_t\int \mathrm{d}z D^0_{\lambda\nu}(x,z)\mathrm{tr}\big[\gamma^\nu\phi^1(t;z,z;\mu,y)\big].$$

It is also possible to start directly from (27) and to write a tree solution of this equation using bare fermion Green functions. Here, the strong field case was treated because it is probably more interesting for applications to solid-state physics.

8 Planar binary trees or planar trees

As a last point, it can be noticed that previous articles have presented general planar trees as the structure adapted to quantum field theory [24]. In fact, planar trees and planar binary trees are equivalent for that purpose. Since the number of planar trees with n vertices is equal to the number of planar binary trees with $2n-1$ vertices [8], there is a bijection Ψ between planar trees and planar binary trees. More precisely, if T_n designates the planar trees with n vertices, there is a bijection Ψ: $T_{n+1}\to Y_n$. In fact, there are $n!$ possible bijections. For instance, if t is a planar tree, we can use the recursive definition $\Psi(\bullet) = \bullet$ and

$$\Psi(B_+(t)) = \Psi(t)\vee\bullet,$$
$$\Psi(B_+(t_1 t_2\dots t_k)) = \Psi(t_1)\vee\Psi(B_+(t_2\dots t_k)),$$

where B_+ is Kreimer's grafting operator ([1,2,5]). The inverse map is given by $\Psi^{-1}(\bullet) = \bullet$ and

$$\Psi^{-1}(t_1\vee t_2) = B_+\big(\Psi^{-1}(t_1)B_-\Psi^{-1}(t_2)\big).$$

If $t_2 = \bullet$, we use the convention that $B_-(\bullet) = 1$ and $B_+\big(\Psi^{-1}(t)1\big) = B_+\big(\Psi^{-1}(t)\big)$.

Such a bijection is also apparent in the existence of two methods for the numerical solution of differential equations on Lie groups: one based on planar trees [25], the other on planar binary trees [26].

Planar binary trees were chosen here because the recursive formulas look simpler and because of the mathematical results of Loday, Frabetti and collaborators.

Furthermore, planar binary trees offer a way to stress the fact that the trees used in this paper are basically different from the rooted trees used in the companion article [5]. To show this more clearly, we can solve the same problem with the methods of the two papers. Consider the equation

$$\psi(x) = \psi_0(x) + \lambda\int \mathrm{d}y G(x,y)(\psi(y))^2$$

Using formula (26) of [5], we obtain

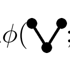
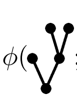
$\psi(x) = \psi_0(x) + \lambda\phi_x(\bullet) + \lambda^2\phi_x($  $) + \cdots$

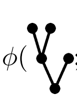
with

$$\phi_x(\bullet) = \int \mathrm{d}y G(x,y)(\psi_0(y))^2,$$

$\phi_x($  $) = 2\int \mathrm{d}y G(x,y)\psi_0(y)\int \mathrm{d}z G(y,z)(\psi_0(z))^2.$

Using planar binary trees, we have now

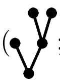
$\psi(x) = \phi(\bullet;x) + \lambda\phi($  $;x) + \lambda^2\phi($  $;x)$

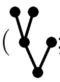
$+\lambda^2\phi($  $;x) + \cdots$

with

$$\phi(\bullet; x) = \psi_0(x),$$

$\phi($  $;x) = \int \mathrm{d}y G(x,y)(\psi_0(y))^2,$

$\phi($  $;x) = \int \mathrm{d}y G(x,y)\psi_0(y) \int \mathrm{d}z G(y,z)(\psi_0(z))^2,$

$\phi($  $;x) = \int \mathrm{d}y G(x,y) \int \mathrm{d}z G(y,z)(\psi_0(z))^2 \psi_0(y).$

If we denote by R_n the set of rooted trees with n vertices, this example demonstrates that a tree t of R_n in the Butcher series is the sum of the contribution of several trees of Y_n in the series over planar binary trees. The difference between the two approaches is due to the fact that planar binary trees allow for the solution of equations involving functional derivatives and non commutative quantities. With this respect, planar binary trees Y_n have an advantage over planar trees T_n: if the problem is commutative, then all planar binary trees corresponding to the same binary tree by permutation of the vertices give the same contribution. If planar trees are used, this property is lost, and trees giving the same contribution can look widely different.

Another difference between the planar binary tree and the rooted tree methods can be shown in the following example:

$$\psi(x) = \psi_0(x) + \lambda \int \mathrm{d}y G(x,y)(\psi(y))^n.$$

If $n > 2$, the recursive solution requires the use of the convolution operation in the case of planar binary trees, whereas some more branches are simply added to the rooted trees for the method of the companion paper.

9 Conclusion

A method was presented to write the solution of some Schwinger equations as a series over planar binary trees. In quantum field theory, it is common to expand over the number of loops or to use integral equations relating, for instance, the full propagator to the full vertex. The first method gives explicit results but becomes very complex, and hundreds of diagrams must be built and calculated after the first few terms of the perturbation expansion. The second method is formally powerful but not very explicit because an n-body Green function is expressed in terms of an unknown $(n+1)$-body Green function. The present approach is a way to mix these two methods to obtain an explicit recursive formula for the propagators and their functional derivatives.

The main point of the method is that explicit recursive expressions can be given for the solution of Schwinger equations. Because of the recursive structure, the results obtained at each step can be reused for the next steps.

The present paper is only a first exploration of the method of series indexed by planar binary trees, and much work remains to be done to investigate its algebraic properties and its applications.

Two kinds of applications were presented here. On the one hand, a series indexed by planar binary trees was given for various physical quantities (full fermion and photon propagators, full two-body Green function), and a method was given to deduce from this a series for vacuum polarization, fermion self-energy and irreducible vertex function. Although the formulas may be a bit cumbersome, they are derived and proved easily. On the other hand, the recursive nature of the terms of the series is well suited to prove properties to all orders of perturbation theory.

The present work can be expanded in various directions. Other field theories can be investigated, as well as many-particle Green functions. For instance, similar formulas have been obtained for the ϕ^3 and ϕ^4 theories, with or without first quantized solutions as a background field. The present treatment was restricted to classical electromagnetic sources; it is worthwhile to study the case of anticommuting fermion sources. Furthermore, planar binary trees could be used to solve the Hedin equation [27] and discuss the GW approximation [28] of solid-state physics.

However, before such developments can take place, it is necessary to investigate the way renormalization can be introduced into the present scheme. This will be the subject of a future publication.

Acknowledgements. I am very grateful to Dirk Kreimer and David Broadhurst for exciting discussions on quantum field theory and renormalization. I thank Alain Connes for his suggestion of concentrating on the calculation of propagators. My warmest thanks go to Ale Frabetti and Jean-Louis Loday for the wonderful day we spent together talking about trees and homology. This is IPGP contribution #1628.

A Appendix

This Appendix contains proofs of some of the statements contained in the text.

A.1 Proof of (8)

Equation (8) will be proved in two steps. Firstly, it will be shown that if $\phi^n(t)$ satisfies (8), then $\delta\phi^n(t)/\delta v = \phi^{n+1}_{\Sigma}(t)$; secondly, that the sum over trees is a solution of (3).

The first step is to show that

$$\frac{\delta\phi^n(t;\{z\}_{1,n})}{\delta v(y)} = \phi^{n+1}_{\Sigma}(t;y,\{z\}_{1,n}). \qquad (29)$$

This will be proved inductively. According to the construction of $\phi^n(\bullet)$ from the initial data A, the relation is true for $t = \bullet$. Now we assume that the relation is true up to trees with $2N-1$ vertices. Let t be a tree with $2N+1$

vertices. The functional derivative of (8) gives us

$$\frac{\delta\phi^n(t;\{z\}_{1,n})}{\delta v(y)} = \sum_{k=0}^{n} F\big(\frac{\delta\phi^k(t_1;\{z\}_{1,k})}{\delta v(y)}, \phi_\Sigma^{n-k+1}(t_2;z,\{z\}_{k+1,n})\big) + \sum_{k=0}^{n} F\big(\phi^k(t_1;\{z\}_{1,k}), \frac{\delta\phi_\Sigma^{n-k+1}(t_2;z,\{z\}_{k+1,n})}{\delta v(y)}\big).$$

Since t_1 and t_2 have less vertices than t, relation (29) is true for them and we obtain

$$\frac{\delta\phi^n(t;\{z\}_{1,n})}{\delta v(y)} = \sum_{k=0}^{n} F\big(\phi_\Sigma^{k+1}(t_1;y,\{z\}_{1,k}), \phi_\Sigma^{n-k+1}(t_2;z,\{z\}_{k+1,n})\big) + \sum_{k=0}^{n} F\big(\phi^k(t_1;\{z\}_{1,k}), \phi_{\Sigma\Sigma}^{n-k+2}(t_2;y,z,\{z\}_{k+1,n})\big), \quad (30)$$

where $\phi_{\Sigma\Sigma}^k(t;\{z\}_{1,k})$ distributes the variables z_1 and z_2 over the k positions $z_1,\dots,z_k$ without changing the order of $z_3,\dots,z_k$. All ways to take two number among k are used, so it is clear that $\phi_{\Sigma\Sigma}^k(t;z_1,z_2,\dots,z_k)$ is symmetric in z_1,z_2.

On the other hand, we know from (8) that

$$\phi^{n+1}(t;y,\{z\}_{1,n}) = F\big(\phi^0(t_1),\phi_\Sigma^{n+2}(t_2;z,y,\{z\}_{1,n})\big) + F\big(\phi^{n+1}(t_1;y,\{z\}_{1,n}),\phi^1(t_2;z)\big) + \sum_{k=1}^{n} F\big(\phi^k(t_1;y,\{z\}_{1,k-1}),\phi_\Sigma^{n-k+2}(t_2;z,\{z\}_{k,n})\big).$$

If we symmetrize y in ϕ^{n+1}, we obtain

$$\phi_\Sigma^{n+1}(t;y,\{z\}_{1,n}) = F\big(\phi^0(t_1),\phi_{\Sigma\Sigma}^{n+2}(t_2;z,y,\{z\}_{1,n})\big) + F\big(\phi_\Sigma^{n+1}(t_1;y,\{z\}_{1,n}),\phi_\Sigma^1(t_2;z)\big) + \sum_{k=1}^{n} F\big(\phi_\Sigma^k(t_1;y,\{z\}_{1,k-1}),\phi_\Sigma^{n-k+2}(t_2;z,\{z\}_{k,n})\big) + \sum_{k=1}^{n} F\big(\phi^k(t_1;\{z\}_{1,k}),\phi_{\Sigma\Sigma}^{n-k+2}(t_2;z,y,\{z\}_{k+1,n})\big).$$

Comparing this with (30), we see that the two expressions are identical, and the property is proved for t.

If we denote by X the (formal) sum $X=\sum_t \phi^0(t)$, then from (29) we obtain $\delta X/\delta v(z) = \sum_t \phi^1(t;z)$. Using (8) we can write

$$X = \sum_t \phi^0(t) = \phi^0(\bullet) + \sum_{t\neq\bullet} F(\phi^0(t_1),\phi^1(t_2;z)).$$

At this point intervenes the essential property (1) that each tree different from $\bullet$ is generated in a unique way by the grafting of two trees t_1 and t_2. The sum over t_1 and t_2, which are the branches of t, can be replaced by an unrestricted sum over all trees t_1 and t_2. Thus we have

$$X = \phi^0(\bullet) + \sum_{t_1,t_2} F\big(\phi^0(t_1),\phi^1(t_2;z)\big) = \phi^0(\bullet) + F\Big(\sum_{t_1}\phi^0(t_1),\sum_{t_2}\phi^1(t_2;z)\Big) = A + F\big(X,\frac{\delta X}{\delta v(z)}\big).$$

A small extension of the previous result is necessary to treat the case of QED. The Schwinger equation is now $X = A + AF(X,\delta X/\delta v(z))$ and the solution is $X = \sum_t \phi(t)$, where $\phi(\bullet)$ is the usual initial data and the recurrence relation becomes

$$\phi^n(t;\{z\}_{1,n}) = \sum_{k=0}^{n}\sum_{k'=0}^{n-k}\phi^k(\bullet;\{z\}_{1,k}) \times F\big(\phi^{k'}(t_1;\{z\}_{k+1,k+k'}),\phi_\Sigma^{n-k-k'+1}(t_2;z,\{z\}_{k+k'+1,n})\big).$$

The reasoning is the same as for the simple Schwinger equation. We start by proving that $\delta\phi^n(t)/\delta v = \phi_\Sigma^{n+1}(t)$. This is done by using the recursive definition of ϕ to write both sides in terms of $\phi(t_1)$ and $\phi(t_2)$. Then we write $X=\sum_t\phi^0(t)$ and we show that, because of the recurrence relation for ϕ, it satisfies $X = A + AF(X,\delta X/\delta v(z))$.

A.2 Proof of (9)

Equation (9) will be proved by two methods. Both are fast and easy. In the first method, we define a Schwinger equation

$$\begin{pmatrix}X\\Z\end{pmatrix} = \begin{pmatrix}A\\0\end{pmatrix} + H\begin{pmatrix}X\\Y\end{pmatrix}, \quad (31)$$

where

$$H\begin{pmatrix}X\\Z\end{pmatrix} = \begin{pmatrix}F\big(X,\frac{\delta X}{\delta v(z)}\big)\\ G\big(X+Z,\frac{\delta X}{\delta v(z)}+\frac{\delta Z}{\delta v(z)}\big)\end{pmatrix}.$$

This is a Schwinger equation whose solution is given by (8), where the map is

$$\begin{pmatrix}\phi\\\psi\end{pmatrix}(t) = \sum_{k=0}^{n}\begin{pmatrix}F\big(\phi^k(t_1),\phi_\Sigma^{n-k+1}(t_2)\big)\\ G\big(\phi^k(t_1)+\psi^k(t_1),\\ \phi_\Sigma^{n-k+1}(t_2)+\psi_\Sigma^{n-k+1}(t_2)\big)\end{pmatrix},$$

which is (9).

The upper component of (31) is the first Schwinger equation $X = A+F(X,\delta X/\delta v(z))$. If we write $Y=X+Z$, the lower component is $Y = X + G(Y,\delta Y/\delta v(z))$. Therefore, $X+Z$ is the solution of the composition of equations. The map corresponding to this solution is $\chi(t) = \phi(t)+\psi(t)$.

The second proof can also be useful. It starts by adding a parameter s to the Schwinger equations: $X(s) = A +$

$sF(X(s), \delta X(s)/\delta v(z))$ and $Y(s) = X(s)+sG(Y(s), \delta Y(s)/\delta v(z))$. We take the nth derivative of Y with respect to s, and we write $Y^{(n)}$ for its value at $s = 0$. The chain rule gives

$$Y^{(n)} = X^{(n)} + n\sum_{k=0}^{n}\binom{n-1}{k}G\big(Y^{(k)}, \frac{\delta Y^{(n-k-1)}}{\delta v(z)}\big).$$

It can be shown recursively that

$$Y^{(n)} = n!\sum_{|t|=n}\big(\phi^0(t) + \psi^0(t)\big),$$

where ψ satisfies (9). The result follows by expanding $X(s)$ and $Y(s)$ as power series of s and taking its value at $s = 1$.

A.3 Proof of (26)

Let $X(s) = 1+\sum_{t\neq \bullet} s^{|t|}\phi(t)$. Let us show that the series for $Y(s) = 1 - 1/X(s)$ is given by $Y(s) = \sum_{t\neq \bullet} s^{|t|}\psi(t)$, where $\psi(t)$ is defined by (26).

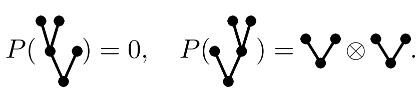
The first step is to prove that, if $P(t)$ is defined by (25), then, with an abuse of notation,

$$P(Y_n) = \sum_{k=1}^{n-1} Y_k \otimes Y_{n-k}, \tag{32}$$

or more precisely

$$\sum_{|t|=n} P(t) = \sum_{k=1}^{n-1}\Big(\sum_{|u|=k} u\Big)\otimes\Big(\sum_{|v|=n-k} v\Big).$$

As usually, this will be proved recursively. The property is true for $n = 2$, because



If this is true up to n, then, from (1)

$$\sum_{|t|=n+1} P(t) = \sum_{k=0}^{n}\sum_{|t_1|=n-k}\sum_{|t_2|=k} P(t_1 \vee t_2).$$

By definition (25),

$$\sum_{|t|=n+1} P(t) = \sum_{k=1}^{n}\sum_{|t_1|=n-k}\sum_{|t_2|=k}(t_1 \vee \bullet)\otimes t_2 + \sum_{k=2}^{n}\sum_{|t_1|=n-k}\sum_{|t_2|=k}\sum_{i=1}^{n(t_2)}(t_1 \vee u_i)\otimes v_i.$$

We use property (32) in the right-hand side and re-order the sum:

$$\sum_{|t|=n+1} P(t) = \sum_{k=1}^{n}\sum_{|t_1|=n-k}\sum_{|t_2|=k}(t_1 \vee \bullet)\otimes t_2 + \sum_{m=1}^{n-1}\sum_{k=m+1}^{n}\sum_{|t_1|=n-k}(t_1 \vee Y_{k-m})\otimes Y_m.$$

In the second term, we use (1) to sum over k by

$$\sum_{k=m+1}^{n} Y_{n-k}\vee Y_{k-m} = Y_{n-m+1} - Y_{n-m}\vee \bullet.$$

Therefore, we have

$$\begin{aligned}\sum_{|t|=n+1} P(t) &= \sum_{k=1}^{n}\sum_{|t_1|=n-k}\sum_{|t_2|=k}(t_1 \vee \bullet)\otimes t_2 \\ &\quad+\sum_{m=1}^{n-1}\sum_{|t_1|=n-m+1}\sum_{|t_2|=m} t_1\otimes t_2 \\ &\quad-\sum_{m=1}^{n-1}\sum_{|t_1|=n-m}\sum_{|t_2|=m}(t_1 \vee \bullet)\otimes t_2 \\ &= \sum_{m=1}^{n}\sum_{|t_1|=n+1-m}\sum_{|t_2|=m} t_1\otimes t_2 \\ &= \sum_{m=1}^{n} Y_{n+1-m}\otimes Y_m.\end{aligned}$$

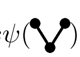
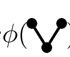
To complete the proof, we start from $X(s) = 1 + \sum_{t\neq \bullet} s^{|t|}\phi(t)$ and we define $Y(s) = \sum_{t\neq \bullet} s^{|t|}\psi(t)$, where $\psi(t)$ is given by $\psi(\bullet) = 0$, $\psi(t) = \phi(t) - \sum\phi(u_i)\psi(v_i)$. Thus

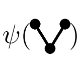
$$\sum_{|t|>1} s^{|t|}\psi(t) = \sum_{|t|>1} s^{|t|}\phi(t) - \sum_{|t|>1} s^{|t|}\sum_{i=1}^{n(t)}\phi(u_i)\psi(v_i).$$

From (32), we see that $|t| = |u_i| + |v_i|$ and

$$\sum_{|t|>1} s^{|t|}\psi(t) = \sum_{|t|>1} s^{|t|}\phi(t) - \Big(\sum_{|u|>0} s^{|u|}\phi(u)\Big)\Big(\sum_{|v|>0} s^{|v|}\psi(v)\Big).$$

From the definition of $X(s)$ and $Y(s)$ we deduce

$Y(s) - s\psi($$) = X(s) - s\phi($$) - 1 - (X(s) - 1)Y(s).$

From the definition of $\psi(t)$ we get $\psi($$) = \phi($$)$, and $Y(s)$ satisfies the equation

$$X(s) - X(s)Y(s) = 1,$$

or $X(s) = 1/(1-Y(s))$. To simplify the notation, we have assumed $\phi(\bullet) = 1$. The general case is proved similarly and leads to (26).

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